natural manner to the relativistic and quantum theory generalizations /8/. In addition, the canonical Hamiltonian formulation of hydrodynamic problems is found to be convenient in the case of numerical calculations /9/.

The author thanks Yu.I. Neimark for his interest.

REFERENCES

1. KOCHIN N.E., KIBEL' I.A. and ROZE N.V., Theoretical Hydromechanics. Pt.1, Moscow, Fizmatgiz, 1963.
2. KOCHIN N.E., Vector Calculus and the Principles of the Tensor Calculus. Moscow, Izd-vo AS SSSR, 1961.
3. KAPLAN S.A., On the conservation of circulation in magnetohydrodynamics. Astron. Zh., 31, 4, 1954.
4. TARAPOV I.E., A variational principle in the hydromechanics of an isotropically magnetizable medium, PMM, 48, 3, 1984.
5. GOLUBINSKII A.I., On the conservation of the generalized velocity circulation in steady flows of an ideal gas. Dokl. AS SSSR, 196, 5, 1971.
6. GOLUBINSKII A.I. and SYCHEV V.V., On some properties of conservation of vortical gas flows. Dokl. AS SSSR, 4, 1977.
7. TER HAAR D., Elements of Hamiltonian Mechanics Oxford, Pergamon Press, 1971.
8. ZAKHAROV V.E., Hamiltonian formalism for waves in non-linear media with dispersion. Izv. vuzov. Radiofizika, 17, 4, 1974.
9. BUNEMAN O., Advantages of Hamiltonian formulations in computer simulations. In: Mathematical methods in hydrodynamics and integrability in dynamical systems. N.Y. Amer. Inst. Phys., 1982.

Translated by L.K.

PMM U.S.S.R.,Vol.50,No.3,pp.391-394,1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

SLOW MOTION OF A PARTICLE IN A WEAKLY ANISOTROPIC VISCOUS FLUID*

V.N. POKROVSKII and A.A. TSKHAI

The problem of the steady flow past a rigid sphere of a linear, homogeneous weakly anisotropic viscous incompressible fluid is studied in the Stokes approximation. The solution is sought using the perturbation method and has the form of an expansion in particular solutions of the Laplace equation in Cartesian coordinates. Expressions for the velocity and pressure fields in the fluid are obtained, as well as for the force acting on the particle.

When studying certain systems such as liquid crystals, we encounter the problem of determining the coefficients of resistance when a particle is in translational and rotational motion through an anisotropic fluid. The simplest case of such a fluid is a linear, homogeneous, viscous anisotropic liquid defines by the equation (see e.g. /1/)

$$\sigma_{ij} = -p\delta_{ij} + \eta_{ijhq}\nabla_q v_h \tag{1}$$
$$(\eta_{ijhq} = \eta_{jihq} = \eta_{ijqh} = \eta_{hqij},\ \nabla_q = \partial/\partial x_q)$$

where σ_{ij} is the stress tensor, p is the pressure, v_h is the velocity and η_{ijhq} is the tensor of viscosity coefficients with the idicated symmetry properties.

We can separate from the tensor of viscosity coefficients η_{ijhq} a part corresponding to an isotropic fluid with viscosity coefficient η

$$\eta_{ijhq} = \eta(\delta_{ih}\delta_{jq} + \delta_{iq}\delta_{jh}) + \xi_{ijhq} \tag{2}$$

Henceforth we shall regard the anisotropic term ξ_{ijhq} as small, and this will make it possible to express the particle resistance coefficients in the form of an expansion in terms of the small parameter ξ_{ijhq}. We will restrict ourselves to determining the first-order correction to the resistance coefficient of a spherical particle in translational motion.

**Prikl.Matem.Mekhan.*,50,3,512-515,1986

Note that the attempt made in /2/ to solve this problem for an arbitrary value of anisotropy has proved to be erroneous (see the appendix).

Let us consider the problem of a spherical particle of radius R moving slowly through an anisotropic liquid at rest, defined by the equation (1), (2). We will obtain the velocity and pressure distribution in the form of a solution to the equations of motion and continuity with the boundary conditions

$$\nabla_j \sigma_{ij} = 0, \quad \nabla_i v_i = 0 \tag{3}$$

We write the expressions for the velocity and pressure up to first-order terms in the small parameter ξ_{ijhq} in the form

$$v_i = v_i^{(0)} + v_i^{(1)}, \quad p = p^{(0)} + p^{(1)} \tag{4}$$

The zero solution corresponds to the Stokes solution and has the form

$$v_i^{(0)} = -\frac{3}{4}\left(\frac{R}{r}\right)[u_i + n_i(u_k n_k)] - \frac{1}{4}\left(\frac{R}{r}\right)^3 [u_i - 3n_i(u_k n_k)] + u_i$$
$$p^{(0)} = -\frac{3}{2}\frac{R}{r^2}\eta n_i u_i$$

We obtain the following system of equations with boundary conditions for the first-order corrections:

$$\nabla_h p^{(1)} = \eta \nabla_s \nabla_s v_h + \xi_{hltm} \nabla_m \nabla_l v_t^{(0)}, \quad \nabla_h v_h^{(1)} = 0 \tag{5}$$
$$r = R, \quad v_i^{(1)} = 0; \quad r \to \infty, \quad v_i^{(1)} = 0 \tag{6}$$

To solve the boundary value problem (5), (6) we follow /3/ and introduce the concept of a multipole $L_{i\ldots k}$ of order θ as a particular solution of the Laplace equation in a Cartesian coordinate system (θ is the number of indices $i\ldots k$)

$$L_{i\ldots k} = \frac{(-1)^\theta}{(2\theta-1)!!}\nabla_k \ldots \nabla_i \left(\frac{1}{r}\right), \quad \nabla_s \nabla_s L_{i\ldots k} = 0$$

The expressions for the multipoles of order $\theta \leqslant 6$ have the form

$$L = \frac{1}{r}, \quad L_i = \frac{x_i}{r^3}, \quad L_{ik} = \frac{x_i x_k}{r^5} - \frac{\delta_{ik}}{3r^3} \tag{7}$$
$$L_{ikl} = \frac{x_i x_k x_l}{r^7} - \frac{(\delta_{\alpha\beta} x_\gamma)_{ikl}}{5r^5}$$
$$L_{iklm} = \frac{x_i x_k x_l x_m}{r^9} - \frac{(\delta_{\alpha\beta} x_\gamma x_\delta)_{iklmn}}{7r^7} + \frac{(\delta_{\alpha\beta}\delta_{\gamma\delta})_{iklm}}{35r^5}$$
$$L_{iklmn} = \frac{x_i x_k x_l x_m x_n}{r^{11}} - \frac{(\delta_{\alpha\beta} x_\gamma x_\delta x_\varepsilon)_{iklmn}}{9r^9} + \frac{(\delta_{\alpha\beta}\delta_{\gamma\beta} x_\varepsilon)_{iklmn}}{63r^7}$$
$$L_{iklmnt} = \frac{x_i x_k x_l x_m x_n x_t}{r^{13}} - \frac{(\delta_{\alpha\beta} x_\gamma x_\delta x_\varepsilon x_\nu)_{iklmnt}}{11r^{11}} + \frac{(\delta_{\alpha\beta}\delta_{\gamma\beta} x_\varepsilon x_\nu)_{iklmnt}}{99r^9} - \frac{(\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\varepsilon\nu})_{iklmnt}}{693r^7}$$

Here and henceforth expressions of the form $(\delta_{\alpha\beta} x_\gamma x_\delta)_{iklm}$ will denote the sum of terms of the same type, in which the Greek indices within the brackets take, consecutively, all values of the Latin indices appearing outside the brackets. The repeated terms are taken into account only once.

The following rules of operation over the multipoles hold:

$$\nabla_q L_{i\ldots k} = -(2\theta+1) L_{i\ldots kq}, \quad x_q L_{i\ldots kq} = \frac{\theta+1}{2\theta+1} L_{i\ldots k}$$
$$\nabla_s \nabla_s r^n L_{i\ldots k} = n(n-2\theta-1) r^{n-2} L_{i\ldots k}$$
$$\nabla_s \nabla_s \left\{x_t r^n L_{i\ldots k} + \frac{2(2\theta+1)}{(n+2)(n-2\theta-1)} r^{n+2} L_{i\ldots kt}\right\} = n(n-2\theta+1) r^{n-2} x_t L_{i\ldots k}$$

The following recurrence relation for the multipoles of order $\theta \leqslant 5$ (the number of indices $\gamma\ldots\delta$ is equal to $I-2$): also holds:

$$x_m L_{i\ldots k} = r^2 L_{i\ldots km} - \frac{2}{(2\theta+1)(2\theta-1)}(\delta_{\alpha\beta} L_{\gamma\ldots\delta m})_{i\ldots k} + \frac{1}{2\theta+1}(\delta_{m\alpha} L_{\gamma\ldots\delta\varepsilon})_{i\ldots k} \tag{8}$$

Applying the operation ∇_h to the first equation of (5) and using the second equation of (5), we obtain Poisson's equation for the pressure correction

$$\nabla_s \nabla_s p^{(1)} = \xi_{hltm} \nabla_h \nabla_m \nabla_l v_t^{(0)} \tag{9}$$
$$v_t^{(0)} = (A_k + B_k r^2) L_{tk} + C_t L + u_t, \quad A_k = \frac{3}{4} R^3 u_k, \quad C_k = -R u_k, \quad B_k = -\frac{3}{4} R u_k$$

Considering Eqs.(5) and (9) together, using the properties of the multipoles shown above and taking into account the boundary conditions (6), we obtain the first velocity and pressure corrections

$$\begin{aligned}
p^{(1)} &= \xi_{nltm} R u_k \Big\{ A L_{tklmn} + B\,(2\delta_{mn} L_{tkl} + \delta_{ln} L_{tkm}) + \\
&C\delta_{kn} L_{tlm} - \frac{9}{140}(\delta_{kl}\delta_{mn} L_t + \delta_{tm}\delta_{ln} L_k) + \frac{12}{35}\delta_{km}\delta_{ln} L_t - \frac{9}{70}\delta_{tl}\delta_{mn} L_k \Big\} \\
A &= \frac{105}{8} R^2 r^2 - \frac{45}{16} r^4 - \frac{189}{16} R^4, \quad B = \frac{1}{2} r^2 - \frac{5}{6} R^2, \\
C &= -\frac{5}{4} r^2 + \frac{25}{12} R^2 \\
v_h^{(1)} &= \frac{R}{\eta}(r^2 - R^2)\,\xi_{nltm} u_k \Big\{ D L_{tklmnh} + E\,(4\delta_{mn} L_{tklh} + 2\delta_{ln} L_{tkmh} + \\
&\delta_{kh} L_{tlmn}) + G\,(\delta_{tk} L_{lmnh} + \delta_{th} L_{klmn}) - \\
&\frac{1}{24} L_{th}\delta_{kl}\delta_{mn} + \frac{1}{6}(\delta_{km}\delta_{ln} L_{th} + \delta_{kl}\delta_{nh} L_{tm}) - \frac{19}{420}\delta_{tl}\delta_{mn} L_{kh} - \\
&\frac{19}{840}\delta_{tm}\delta_{ln} L_{kh} + \frac{5}{42}\delta_{lm}\delta_{nh} L_{tk} + \frac{5}{84}\delta_{ln}\delta_{mh} L_{tk} - \\
&\frac{1}{21}\delta_{kh}\delta_{mn} L_{tl} - \frac{5}{12}\delta_{km}\delta_{nh} L_{tl} - \frac{1}{42}\delta_{kh}\delta_{ln} L_{tm} \Big\} \\
D &= -\frac{15}{32} r^4 + \frac{45}{16} r^2 R^2 - \frac{99}{32} R^4, \quad E = \frac{9}{176} r^2 - \frac{21}{176} R^2, \\
G &= -\frac{63}{176} r^2 + \frac{147}{176} R^2
\end{aligned} \tag{10}$$

The force acting on the particle is found from the formula

$$F_i = \iint_S \sigma_{ik} n_k d_s$$

where n_k is the unit normal to the particle surface S.

The stress tensor and the velocity and pressure field strength (4) are given, up to first-order terms in ξ_{ijhq}, in the form

$$\sigma_{ij} = \sigma_{ij}^{(0)} + \sigma_{ij}^{(1)}, \quad F_i = F_i^{(0)} + F_i^{(1)}$$

According to the Stokes we have, in the zeroth approximation,

$$F_i^{(0)} = 6\pi\eta R u_i \tag{11}$$

The first correction to the force acting on the particle from the liquid, caused by the anisotropy of the medium, is found from the formula

$$F_i^{(1)} = \iint_S \sigma_{ij}^{(1)} n_j|_{r=R}\, ds = \iint_S [-p^{(1)}\delta_{ij} + \eta(\nabla_j v_i^{(1)} + \nabla_i v_j^{(1)}) + \xi_{ijhq}\nabla_q v_h^{(0)}]\,|_{r=R} n_j\, ds$$

Using the expressions for the velocity and pressure distribution and taking into account the fact that

$$\iint_S x_i x_k\, ds = \frac{4}{3}\pi R^4 \delta_{ik}, \quad \iint_S x_i x_k x_l x_m\, ds = \frac{4}{15}\pi R^6 (\delta_{\alpha\beta}\delta_{\gamma\delta})_{iklm}$$

$$\iint_S x_i x_k x_l x_m x_n x_t\, ds = \frac{4}{105}\pi R^8 (\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\varepsilon\nu})_{iklmnt}$$

we obtain the correction for the reaction force

$$F_i^{(1)} = \frac{3}{35}\pi R\,(15\delta_{tk}\delta_{lm}\delta_{ni} - 10\delta_{tk}\delta_{ln}\delta_{mi} + 2\delta_{tl}\delta_{ki}\delta_{mn} + \delta_{tm}\delta_{ki}\delta_{ln})\,\xi_{nltm} u_k \tag{12}$$

It can be shown that the part of the reaction force depending on the rotational motion of the sphere in an anisotropic liquid is equal to zero. Therefore, the total force of reaction of the sphere is reduced to its force of reaction in its translational motion.

Appendix. On the erroneous results of /2/. In /2/ an attempt was made to solve problem (1), (3) by inroducing new characteristic variables, namely the velocity field tensor V_{is}' and pressure field vector P_s', as follows:

$$v_i = V_{is}' u_s, \quad p\delta_{ij} = {}^1\!/_2 \eta_{ijlm}\delta_{lm} P_s' u_s \tag{A.1}$$

where the translational velocity of the particle u_s is an arbitrary vector. Substituting (A.1) into (1), (3) we find that the characteristic quantities (V_{is}', P_s') must satisfy the following system of equations and boundary conditions:

$$\nabla_j \sigma_{ij} = \frac{1}{2}\eta_{ijlm} u_s \nabla_j(-\delta_{lm} P_s' + \nabla_m V_{ls}' + \nabla_l V_{ms}') = 0 \tag{A.2}$$

$$\nabla_i V_{il}' = 0; \quad r = a, \quad V_{ij}' = \delta_{ij}; \quad r \to \infty, \quad V_{ij}' \to 0 \tag{A.3}$$

We find, however, that the expressions given in /2/ for the characteristic quantities

$$V_{ij}' = \frac{3}{4}\frac{a}{r}(n_i n_j + \delta_{ij}) - \frac{3}{4}\left(\frac{a}{r}\right)^3\left(n_i n_j - \frac{1}{3}\delta_{ij}\right) \quad (A.4)$$

$$P_i' = \frac{3}{2}\frac{a}{r^2}n_i, \quad n_i = \frac{x_i}{r}$$

do not satisfy Eqs.(A.2). Therefore the expressions given for the coefficients of resistance of a spherical particle are incorrect.

The error in /2/ is caused by the fact that instead of Eq.(A.2) the author used

$$\nabla_l P_s' = \nabla_j \nabla_j V_{ls}' \quad (A.5)$$

which were obtained as follows. In order for the pair (V_{is}', P_s') to be a solution of (A.2), it is sufficient for that pair to be a solution of the equation $\nabla_j(-\delta_{lm}P_s' + \nabla_m V_{ls}' + \nabla_l V_{ms}') = 0$, or in expanded form, to

$$\begin{aligned}
&j = m = 1, \quad -\delta_{l1}\nabla_1 P_s' + \nabla_1\nabla_1 V_{ls}' + \nabla_l\nabla_1 V_{1s}' = 0 \\
&j = m = 2, \quad -\delta_{l2}\nabla_2 P_s' + \nabla_2\nabla_2 V_{ls}' + \nabla_l\nabla_2 V_{2s}' = 0 \\
&j = m = 3, \quad -\delta_{l3}\nabla_3 P_s' + \nabla_3\nabla_3 V_{ls}' + \nabla_l\nabla_3 V_{3s}' = 0 \\
&j = 1, \quad m = 2, \quad -\delta_{l2}\nabla_1 P_s' + \nabla_1\nabla_2 V_{ls}' + \nabla_l\nabla_1 V_{2s}' = 0 \\
&j = 1, \quad m = 3, \quad -\delta_{l3}\nabla_1 P_s' + \nabla_1\nabla_3 V_{ls}' + \nabla_l\nabla_1 V_{3s}' = 0 \\
&j = 2, \quad m = 2, \quad -\delta_{l1}\nabla_2 P_s' + \nabla_1\nabla_2 V_{ls}' + \nabla_l\nabla_2 V_{1s}' = 0 \text{ etc.}
\end{aligned} \quad (A.6)$$

In /2/ the first three equations of (A.6) were combined to obtain, naturally, (A.5), and the remaining equations of (A.6) were neglected. The present discussion shows, however, that the system (A.3), (A.5) is not equivalent to the system (A.2), (A.3).

We note that in the special case of isotropic viscosity

$$\eta_{ijlm} = \eta(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})$$

the relations (A.1) become $v_i = V_{is}'u_s$, $p = P_s'u_s\eta$, equations (A.2) reduce to Eqs.(A.5) and the problem, as well as the method of solving it in /2/, become identical with the results in /4/.

REFERENCES

1. LANDAU L.D. and LIFSHITZ E.M., Mechanics of Continuous Media. Moscow, Gostekhizdat, 1953.
2. VOLKOV V.S., Slow motions of a sphere in an anisotropic, viscoelastic fluid. PMM, 46, 2, 1982.
3. MIROSHNIKOV V.A., Magnetohydrodynamic flows past a stationary sphere. Magnit. gidrodinamika, 3, 1980.
4. HAPPEL J. and BRENNER H., Low Reynolds Numbers Hydrodynamics. N.Y. Prentice Hall, 1965.

Translated by L.K.

PMM U.S.S.R., Vol.50.No.3, pp.394-399, 1986
Printed in Great Britain
0021-8928/86 $10.00+0.00

ON THE STABILITY OF A VAPOUR-LIQUID MEDIUM CONTAINING BUBBLES*

V.SH. SHAGAPOV

A problem of the stability of a vapour-liquid medium containing bubbles is investigated. It is shown that since the surface tension and phase transitions act simultaneously, a range of values of the parameters of the vapour-liquid and vapour-gas-liquid media containing bubbles exists, for which the equilibrium state is unstable. The effect of various parameters of the two-phase medium, such as the volume content of the bubbles, the mass content of the gas and the degree of dispersion of the medium, on the increment characterizing the rate of development of the instability, is analysed.

1. Fundamental equations. Let us consider the propagation of small perturbations through a polydisperse mixture of liquid and bubbles of $m-1$ kinds, under the usual assumptions made for two-phase media. Moreover, we shall assume that the gaseous phase consists of the vapour from the liquid phase, and some "inert" gas which takes no part in the process

**Prikl.Matem.Mekhan.*, 50, 3, 516-521, 1986.